Int J Thermophys (2010) 31:648–662
DOI 10.1007/s10765-010-0762-4

A Generalized Thermoelasticity Problem for a Half-Space with Heat Sources and Body Forces

Nasser Mohamed El-Maghraby

Received: 15 October 2009 / Accepted: 18 May 2010 / Published online: 9 June 2010

Abstract In this work, a two-dimensional problem of distribution of thermal stresses and temperature in a linear theory of a generalized thermoelastic half-space under the action of a body force and subjected to a thermal shock on the bounding plane is considered. Heat sources permeate the medium. The problem is in the context of the theory of generalized thermoelasticity with one relaxation time. Laplace and exponential Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach. The inverse double transform is evaluated numerically. Numerical results are obtained and represented graphically.

Keywords Body forces · Exponential Fourier transform · Generalized thermoelasticity · Half space · Heat sources · Laplace transform

1 Introduction

Thermal stresses arise in many familiar areas and have been a subject of interest. They are frequently an important factor in determining material life. In coupled theory of thermoelasticity, if an elastic continuum is subjected to a mechanical or thermal disturbance, the effect of the disturbance will be felt instantaneously in both the fields as governing equations are coupled. Physically, this means that a portion of the disturbance has an infinite velocity of propagation. Such behavior is physically inadmissible. Lord and Shulman [1] have presented the generalized theory of thermoelasticity to eliminate this paradox by employing a more general functional relation between heat flow and the temperature gradient. They considered the relaxation time

N. M. El-Maghraby (✉)
Department of Basic and Applied Science, Arab Academy for Science and Technology, Alexandria, Egypt
e-mail: kokimemo@gmail.com

as the time lag needed for the onset of a thermal wave. Dhaliwal and Sherief [2] extended this theory to include the anisotropic case. The uniqueness of a solution for this theory was proved under different conditions by Ignaczak [3], by Sherief and Dhaliwal [4], and by Sherief [5]. The state space approach to this theory was developed by Anwar and Sherief [6] and by Sherief [7] for one-dimensional problems and by Sherief and Anwar [8] for two-dimensional problems. The fundamental solution for this theory was obtained by Sherief [9] and by Sherief and Anwar [10]. Sherief and El-Maghraby [11] have solved a problem for a penny-shaped crack in an infinite thermoelastic solid and have obtained the solution for a Mode-I crack problem in an infinite space in [12]. Sherief and Helmy have solved a two-dimensional problem in [13]. A two-dimensional problem for a half-space and for a thick plate with heat sources has been solved by El-Maghraby [14,15]. A one-dimensional problem for a half-space under the action of a body force has been solved by Heba Saleh [16]. A two-dimensional problem for a thick plate under the action of a body force in two relaxation times has been solved by El-Maghraby [17]. Mallik and Kanoria [18] have solved a two-dimensional problem for a transversely isotropic generalized thick plate with spatially varying heat.

In this work, we consider a two-dimensional problem of distribution of thermal stresses and temperature in a linear theory of a generalized thermoelastic half-space under the action of a body force and subjected to a thermal shock on the bounding plane. Heat sources permeate the medium. The problem is in the context of the theory of generalized thermoelasticity with one relaxation time. Laplace and exponential Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach. The inverse double transform is evaluated numerically. Numerical results are obtained and represented graphically.

2 Formulation of the Problem

We consider a homogeneous isotropic thermoelastic solid occupying the half-space $x \geq 0$. The x-axis is taken perpendicular to the bounding plane pointing inward. We also assume that the initial state of the medium is quiescent. The bounding surface is kept at a given temperature that is a function of y. The displacement vector thus has the form,

$$\underline{\boldsymbol{u}} = (u, v, 0).$$

The equations of motion can be written as [16]

$$\rho \frac{\partial^2 \underline{\boldsymbol{u}}}{\partial t^2} = (\lambda + \mu)\text{grad}\, e + \mu \nabla \underline{\boldsymbol{u}} - \gamma\, \text{grad}\, T + \rho F_x, \tag{1}$$

$$\rho \frac{\partial^2 \underline{\boldsymbol{v}}}{\partial t^2} = (\lambda + \mu)\text{grad}\, e + \mu \nabla^2 \underline{\boldsymbol{v}} - \gamma\, \text{grad}\, T + \rho F_y. \tag{2}$$

The generalized equation of heat conduction has the form [14],

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\rho c_E T + \gamma T_0 e) - \rho \left(1 + \tau_0 \frac{\partial}{\partial t}\right) Q \tag{3}$$

In the above equations, T is the absolute temperature and e is the cubical dilatation given by the relation [14],

$$e = \operatorname{div} \underline{\boldsymbol{u}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{4}$$

In the preceding equations, ρ is the density, λ and μ are Lamé's constants, k is the thermal conductivity, γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t being the coefficient of linear thermal expansion, T_0 is a reference temperature assumed to be such that $|(T - T_0)/T_0| \ll 1$, c_E is the specific heat at constant strain, and τ_0 is a constant with the dimensions of time that acts as a relaxation time. A dot denotes differentiation with respect to time, whereas ∇^2 is Laplace's operator, given in our case by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The components of the stress tensor are given by

$$\sigma_{ij} = 2\mu e_{ij} + \left[\lambda e - \gamma (T - T_0)\right] \delta_{ij}. \tag{5}$$

We shall use the following non-dimensional variables:

$$\begin{aligned}
&x' = c_1 \eta x \quad && y' = c_1 \eta y \quad && u' = c_1 \eta u \quad && v' = c_1 \eta v \\
&t' = c_1^2 \eta t \quad && \tau_0' = c_1^2 \eta \tau_0 \quad && F_x' = \frac{\rho F_x}{(\lambda + 2\mu) c_1 \eta} \quad && F_y' = \frac{\rho F_y}{(\lambda + 2\mu) c_1 \eta} \\
&\sigma_{ij}' = \frac{\sigma_{ij}}{\mu} \quad && \theta = \frac{\gamma (T - T_0)}{(\lambda + 2\mu)} \quad && Q' = \frac{\rho \gamma Q}{k\, c_1^2 \eta^2 (\lambda + 2\mu)},
\end{aligned}$$

where $\eta = \frac{\rho c_E}{k}$ and $c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$. c_1 is the speed of propagation of isothermal longitudinal elastic waves. Using the above dimensionless variables, the governing equations take the form (dropping the primes for convenience),

$$\beta^2 \frac{\partial^2 u}{\partial t^2} = (\beta^2 - 1) \frac{\partial e}{\partial x} + \nabla^2 u - \beta^2 \frac{\partial \theta}{\partial x} + \beta^2 F_x \tag{6}$$

$$\beta^2 \frac{\partial^2 v}{\partial t^2} = (\beta^2 - 1) \frac{\partial e}{\partial y} + \nabla^2 v - \beta^2 \frac{\partial \theta}{\partial y} + \beta^2 F_y \tag{7}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)(\theta + \varepsilon e) - \left(1 + \tau_0 \frac{\partial}{\partial t}\right) Q \tag{8}$$

while the constitutive relation Eq. 5, becomes

$$\sigma_{xx} = 2\frac{\partial u}{\partial x} + (\beta^2 - 2)e - \beta^2\theta \tag{9a}$$

$$\sigma_{yy} = 2\frac{\partial v}{\partial y} + (\beta^2 - 2)e - \beta^2\theta \tag{9b}$$

$$\sigma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{9c}$$

In the above equations, we have

$$\beta^2 = \frac{\lambda + 2\mu}{\mu}, \quad \varepsilon = \frac{T_0\gamma^2}{\rho c_E(\lambda + 2\mu)}$$

Combining Eqs. 6 and 7, we obtain upon using Eq. 4,

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) e = \nabla^2\theta - \operatorname{div} \underline{F} \tag{10}$$

The boundary conditions are taken as

$$\theta(0, y, t) = \theta_0 H(t)\, H(c - |y|) \tag{11a}$$

$$\sigma_{xx}(0, y, t) = 0 \tag{11b}$$

$$\sigma_{xy}(0, y, t) = 0 \tag{11c}$$

In Eq. 11, θ_0 and $\boldsymbol{c}$ are constants while $H(.)$ is the Heaviside unit step function. Thus, the surface $\boldsymbol{x} = 0$ is heated on a band of width $2c$ around the y-axis, while the rest of the surface is kept at the reference temperature $T_0(\theta = 0)$.

3 Solution in the Laplace Transform Domain

Applying the Laplace transform defined by the relation,

$$\bar{f}(x, y, s) = \mathcal{L}[f(x, y, t)] = \int_0^\infty \mathrm{e}^{-st} f(x, y, t)\,\mathrm{d}t$$

to both sides of Eqs. 6–10, we obtain

$$(\nabla^2 - \beta^2 s^2)\bar{u} = (1 - \beta^2)\frac{\partial \bar{e}}{\partial x} + \beta^2\frac{\partial \bar{\theta}}{\partial x} - \beta^2 \bar{F}_x \tag{12}$$

$$(\nabla^2 - \beta^2 s^2)\bar{v} = (1 - \beta^2)\frac{\partial \bar{e}}{\partial y} + \beta^2\frac{\partial \bar{\theta}}{\partial y} - \beta^2 \bar{F}_y \tag{13}$$

$$(\nabla^2 - s - \tau_0 s^2)\,\bar{\theta} = \varepsilon(s + \tau_0 s^2)\,\bar{e} - (1 + \tau_0 s)\bar{Q} \tag{14}$$

$$(\nabla^2 - s^2)\bar{e} = \nabla^2\bar{\theta} - \text{div}\,\underline{\bar{F}} \tag{15}$$

$$\bar{\sigma}_{xx} = 2\frac{\partial \bar{u}}{\partial x} + (\beta^2 - 2)\bar{e} - \beta^2\bar{\theta} \tag{16a}$$

$$\bar{\sigma}_{yy} = 2\frac{\partial \bar{v}}{\partial z} + (\beta^2 - 2)\bar{e} - \beta^2\bar{\theta} \tag{16b}$$

$$\bar{\sigma}_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \tag{16c}$$

The boundary conditions, Eq. 11, in the transformed domain, take the form,

$$\bar{\theta}(0, y, s) = \frac{\theta_0}{s} H(c - |y|) \tag{17a}$$

$$\bar{\sigma}_{xx}(0, y, s) = 0 \tag{17b}$$

$$\bar{\sigma}_{xy}(0, y, s) = 0 \tag{17c}$$

Eliminating $\bar{e}$ between Eqs. 14 and 15, we get

$$\begin{aligned}\left[\nabla^4 - \left(s^2 + s(1+\varepsilon)(1+\tau_0 s)\right)\nabla^2 + s^3(1+\tau_0 s)\right]\bar{\theta} \\ = -(1+\tau_0 s)\left(\varepsilon\, s\, \text{div}\,\underline{\bar{F}} + (\nabla^2 - s^2)\bar{Q}\right)\end{aligned} \tag{18}$$

The above equation can be factorized as

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right)\bar{\theta} = -(1+\tau_0 s)\left(\varepsilon\, s\, \text{div}\,\underline{\bar{F}} + (\nabla^2 - s^2)\bar{Q}\right) \tag{19}$$

where k_1^2 and k_2^2 are the roots of the characteristic equation,

$$k^4 - \left(s^2 + s(1+\tau_0 s)(1+\varepsilon)\right)k^2 + s^3(1+\tau_0 s) = 0 \tag{20}$$

The solution of Eq. 19 can be written in the form,

$$\bar{\theta} = \bar{\theta}_1 + \bar{\theta}_2 + \bar{\theta}_p$$

where $\bar{\theta}_i$ is a solution of the homogenous equation,

$$(\nabla^2 - k_i^2)\,\bar{\theta}_i = 0, \quad i = 1, 2 \tag{21}$$

and $\bar{\theta}_p$ is a particular solution of Eq. 19.

In order to solve the problem, we shall use the exponential Fourier transform with respect to y. This transform of a function $\bar{f}(x, y, s)$ is defined by the relation [19,20],

$$\bar{f}^*(x, q, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(x, y, s)\, e^{-iqy}\, dy$$

The inverse transform is given by the relation [19,20],

$$\bar{f}(x, y, s) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \bar{f}^*(x, q, s) \mathrm{e}^{\mathrm{i}qy} \mathrm{d}q$$

Applying the Fourier exponential transform to both sides of Eq. 21, we obtain

$$\left(\mathcal{D}^2 - \mu_i^2\right) \bar{\theta}_i^* = 0, \quad i = 1, 2 \tag{22}$$

where $\mathcal{D} = \frac{\partial}{\partial x}$, $\mu_i^2 = k_i^2 + q^2$, $i = 1, 2$.

The solution of Eq. 22, that is bounded as $x \to \infty$, can be written as

$$\bar{\theta}_i^* = A_i (k_i^2 - s^2)\, \mathrm{e}^{-\mu_i x}, \quad i = 1, 2 \tag{23a}$$

where $A_i = A_i(q, s)$, $i = 1, 2$, are parameters depending on q and s.

From now on, we shall take the body force in the form,

$$F_x(x, y, t) = \frac{H(t)H(x - a)}{(1 + y^2)}, \quad F_y(x, y, t) = 0$$

where a is a constant.

Thus,

$$\bar{F}_x(x, y, s) = \frac{H(a - x)}{s\,(1 + y^2)}, \quad \bar{F}_y(x, y, s) = 0$$

and

$$\bar{F}_x^*(x, q, s) = \sqrt{\frac{\pi}{2}} \frac{H(x - a)\mathrm{e}^{-a|q|}}{s}, \quad \bar{F}_y^*(x, q, s) = 0$$

From now on, we shall take the heat source in the form,

$$Q(x, y, t) = \frac{H(t)\, \mathrm{e}^{-bx}}{(1 + y^2)}$$

where b is a constant.

The heat source in the Laplace Fourier transformed domain can be written as

$$\bar{Q}^*(x, q, s) = \sqrt{\frac{\pi}{2}} \frac{\mathrm{e}^{-bx} \mathrm{e}^{-|q|}}{s}$$

The particular solution of the Fourier transform of Eq. 19 can be written in the form,

$$\bar{\theta}_p^* = -\sqrt{\frac{\pi}{2}} \frac{(1 + \tau_0 s)(b^2 - \mu^2)\mathrm{e}^{-|q|}\mathrm{e}^{-bx}}{s(b^2 - \mu_1^2)(b^2 - \mu_2^2)} \tag{23b}$$

Using Eq. 23, the solution $\bar{\theta}^*$ takes the form,

$$\bar{\theta}^* = \sum_{i=1}^{2} A_i (k_i^2 - s^2) \mathrm{e}^{-\mu_i x} - \sqrt{\frac{\pi}{2}} \frac{(1+\tau_0 s)(b^2 - \mu^2)\mathrm{e}^{-|q|}\mathrm{e}^{-bx}}{s(b^2-\mu_1^2)(b^2-\mu_2^2)} \tag{24}$$

By the same reasoning we can write

$$\bar{e}^* = \sum_{i=1}^{2} A_i k_i^2 \mathrm{e}^{-\mu_i x} - \sqrt{\frac{\pi}{2}} \left[\frac{(b^2-q^2)(1+\tau_0 s)}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right] \mathrm{e}^{-|q|}\mathrm{e}^{-bx} \tag{25}$$

where $\mu = \sqrt{q^2+s^2}$.

Taking the Fourier transform of Eq. 12, we get

$$\left(\mathcal{D}^2 - q^2 - \beta^2 s^2\right) \bar{u}^* = (1-\beta^2)\mathcal{D}\,\bar{\mathrm{e}}^* + \beta^2 \mathcal{D}\,\bar{\theta}^* - \beta^2 \bar{F}_x^* \tag{26}$$

Substituting from Eqs. 24 and 25 into the right-hand side of Eq. 26, we obtain the following equation satisfied by $\bar{u}^*$:

$$\begin{aligned}\left(\mathcal{D}^2 - q^2 - \beta^2 s^2\right) \bar{u}^* &= \sum_{i=1}^{2} \left(\beta^2 s^2 - k_i^2\right) \mu_i A_i \mathrm{e}^{-\mu_i x} \\ &\quad - \sqrt{\frac{\pi}{2}} \left[\frac{\beta^2 H(x-a)}{s} + \frac{b(1+\tau_0 s)\,(q^2+\beta^2 s^2 - b^2)\,\mathrm{e}^{-bx}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right] \mathrm{e}^{-|q|}\end{aligned} \tag{27}$$

The solution $\bar{u}^*$ of Eq. 27 has the form,

$$\begin{aligned}\bar{u}^* &= B\mathrm{e}^{-\mu_3 x} - \sum_{i=1}^{2} A_i \mu_i \mathrm{e}^{-\mu_i x} \\ &\quad + \sqrt{\frac{\pi}{2}} \left[\frac{\beta^2 H(x-a)}{s\,\mu_3^2} + \frac{b(1+\tau_0 s)\mathrm{e}^{-bx}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right] \mathrm{e}^{-|q|}\end{aligned} \tag{28}$$

where $\mu_3 = \sqrt{q^2+\beta^2 s^2}$ and $B = B(q, s)$ is a parameter depending on q and s.

Applying the Laplace transform and then the exponential Fourier transform with respect to the variable y to both sides of Eq. 4, we get

$$\bar{v}^* = \frac{1}{\mathrm{i}q} \left(\bar{e}^* - \mathcal{D}\,\bar{u}^*\right) \tag{29}$$

Substituting from Eqs. 25 and 28 into the right-hand side of Eq. 29, we obtain

$$\bar{v}^* = \mathrm{i}q \left\{ \sum_{i=1}^{2} A_i \mathrm{e}^{-\mu_i x} - \sqrt{\frac{\pi}{2}} \left[\frac{(1+\tau_0 s)\mathrm{e}^{-|q|}\mathrm{e}^{-bx}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right] - \frac{B\mu_3 \mathrm{e}^{-\mu_3 x}}{q^2} \right\} \tag{30}$$

Taking the Laplace and the exponential Fourier transforms of both sides of Eqs. 9a and 9c, we obtain upon using Eqs. 24, 25, 28, and 30, the transforms of the components of the stress tensor in the form,

$$\bar{\sigma}^*_{xx} = -2B\mu_3 \mathrm{e}^{-\mu_3 x} + \left(\beta^2 s^2 + 2q^2\right) \left(\sum_{i=1}^{2} A_i \mathrm{e}^{-\mu_i x} - \frac{(1+\tau_0 s)\mathrm{e}^{-|q|}\mathrm{e}^{-bx}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right) \tag{31}$$

$$\begin{aligned}\bar{\sigma}^*_{xy} = \mathrm{i}q \Bigg\{ & \frac{(\beta^2 s^2 + 2q^2)}{q^2} B\mathrm{e}^{-\mu_3 x} - 2\sum_{i=1}^{2} A_i \mu_i \mathrm{e}^{-\mu_i x} \\ & + \sqrt{\frac{\pi}{2}} \left[\frac{\beta^2 H(x-a)}{s\,\mu_3^2} + \frac{2b(1+\tau_0 s)\mathrm{e}^{-bx}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right] \mathrm{e}^{-|q|} \Bigg\}\end{aligned} \tag{32}$$

The boundary conditions, Eq. 17, in the transformed Fourier domain, take the form,

$$\bar{\theta}^*(0,q,s) = \sqrt{\frac{\pi}{2}} \frac{\theta_0}{s} \frac{\sin qc}{q} \left(1 - \mathrm{i}\pi q \delta(q)\right) \tag{33a}$$

where $\delta(.)$ is the Dirac delta function.

$$\bar{\sigma}^*_{xx}(0,q,s) = 0 \tag{33b}$$

$$\bar{\sigma}^*_{xy}(0,q,s) = 0 \tag{33c}$$

Using the boundary conditions, Eqs. 33a, 33b, and 33c, to evaluate the parameters A_1, A_2, and B, we obtain the three equations,

$$\begin{aligned} & A_1(k_1^2 - s^2) + A_2(k_2^2 - s^2) \\ & \quad = \sqrt{\frac{\pi}{2}} \left(\frac{\theta_0\,(1 - \mathrm{i}\pi q\delta(q)) \sin qc}{qs} + \frac{(1+\tau_0 s)\,(b^2-\mu^2)\,\mathrm{e}^{-|q|}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \right)\end{aligned} \tag{34}$$

$$\left(2q^2 + \beta^2 s^2\right)[A_1 + A_2] - 2\mu_3 B = \sqrt{\frac{\pi}{2}} \frac{(1+\tau_0 s)(\beta^2 s^2 + 2q^2)\mathrm{e}^{-|q|}}{s(b^2-\mu_1^2)(b^2-\mu_2^2)} \tag{35}$$

$$2A_1\mu_1 + 2A_2\mu_2 - \frac{\beta^2 s^2 + 2q^2}{q^2} B = -\sqrt{\frac{\pi}{2}} \frac{2b(1+\tau_0 s)\mathrm{e}^{-a|q|}}{s\,(b^2-\mu_1^2)(b^2-\mu_2^2)} \tag{36}$$

If a solution (A_1, A_2, B) to Eqs. 34–36 is found, a formal solution to the problem in the transformed domain can be obtained.

4 Inversion of the Double Transform

We shall now outline the numerical inversion method used to find the solution in the physical domain. Let $\bar{f}^*(x, q, s)$ be the Laplace–Fourier transform of a function $f(x, y, t)$. The complex inversion formula for Laplace transforms can be written as [19]

$$f^*(x, q, t) = \frac{1}{2\mathrm{i}\pi} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{st} \bar{f}^*(x, q, s)\mathrm{d}s$$

where c is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}^*(x, q, s)$. Taking $s = c + \mathrm{i}r$, the above integral takes the form,

$$f^*(x, q, t) = \frac{\mathrm{e}^{ct}}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{itr}} \bar{f}^*(x, q, c + \mathrm{i}r)\mathrm{d}r$$

Expanding the function $h(x, q, t) = \mathrm{e}^{-ct} f^*(x, q, t)$ in a Fourier series in the interval $0 \leq t \leq 2L$, for fixed x, q, we obtain the approximate formula [21],

$$f^*(x, q, t) = f^*_\infty(x, q, t) + E_\mathrm{D}$$

where

$$f^*_\infty(x, q, t) = \frac{1}{2}c_0(x, q, t) + \sum_{k=1}^{\infty} c_k(x, q, t) \quad \text{for } 0 \leq t \leq 2L \tag{37}$$

and

$$c_k(x, q, t) = \frac{\mathrm{e}^{ct}}{L}\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}k\pi t/L}\ \bar{f}^*(x, q, c + \mathrm{i}k\pi/L)\right], \quad k = 0, 1, 2, \ldots \tag{38}$$

E_D, the discretization error, can be made arbitrarily small by choosing the constant c sufficiently large [21].

We note that the number L is fixed during computation. Its optimal value that ensures fast convergence is approximately 1.5 t.

Since the infinite series in Eq. 37 can only be summed up to a finite number N of terms, the approximate value of $f^*(x, q, t)$ becomes

$$f^*_N(x, q, t) = \frac{1}{2}c_0(x, q, t) + \sum_{k=1}^{N} c_k(x, q, t) \quad \text{for } 0 \leq t \leq 2L \tag{39}$$

Using the above formula to evaluate $f^*(x, q, t)$, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the 'Korrecktur' method [21] is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function $f^*(x, q, t)$:

$$f^*(x, q, t) = f^*_\infty(x, q, t) - \mathrm{e}^{-2cL} f^*_\infty(x, q, 2L + t) + E'_\mathrm{D}$$

where the discretization error $\left|E'_\mathrm{D}\right| \ll |E_\mathrm{D}|$ [21].

Thus, the approximate value of $f^*(x, q, t)$ becomes

$$f^*_{NK}(x, q, t) = f^*_N(x, q, t) - \mathrm{e}^{-2cL} f^*_{N'}(x, q, 2L + t) \tag{40}$$

N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. 39. Let N be an odd natural number, and let

$$s_m(x, q, t) = \sum_{k=1}^{m} c_k(x, q, t)$$

be the sequence of partial sums of Eq. 39. We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m}), \quad p = 1, 2, 3, \ldots$$

It can be shown that [21] the sequence

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$$

converges to $f^*(x, q, t) + E_\mathrm{D} - c_0/2$ faster than the sequence of partial sums,

$$s_m, \quad m = 1, 2, 3, \ldots$$

The actual procedure used to invert the Laplace transforms consists of using Eq. 40 together with the ε-algorithm. The values of c and L are chosen according to the criteria outlined in [21].

Next the inverse exponential Fourier transform of the function $f^*(x, q, t)$ is obtained from the formula,

$$f(x, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}qy} f^*(x, q, t)\,\mathrm{d}q$$

Table 1 Properties

$k = 386\ \mathrm{J \cdot K^{-1} \cdot m^{-1} \cdot s^{-1}}$	$\alpha_t = 1.78\times10^{-5}\ \mathrm{K^{-1}}$	$c_E = 383.1\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$	$\eta = 8886.73\ \mathrm{s \cdot m^{-2}}$
$\mu = 3.86 \times 10^{10}\ \mathrm{N \cdot m^{-2}}$	$\lambda = 7.76\times10^{10}\ \mathrm{N \cdot m^{-2}}$	$\rho = 8954\ \mathrm{kg \cdot m^{-3}}$	$c_1 = 4.158\times10^{3}\mathrm{m \cdot s^{-1}}$
$\tau_0 = 0.02\ \mathrm{s}$	$T_0 = 293\ \mathrm{K}$	$\varepsilon = 0.0168\ \mathrm{N \cdot m \cdot J^{-1}}$	$\beta^2 = 4$
$\theta_0 = 1$	$a = 1$	$b = 1$	$c = 1$

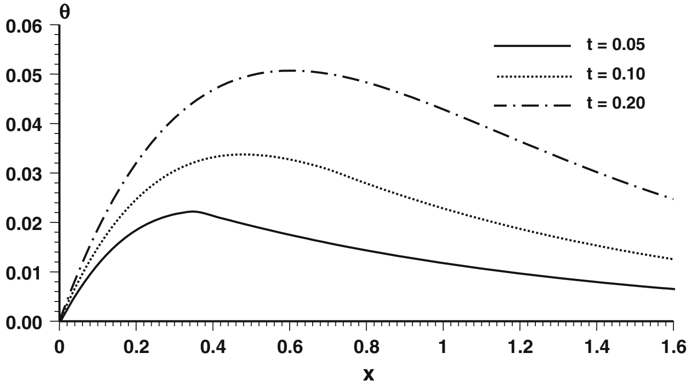

Fig. 1 Temperature distribution in the absence of thermal shock

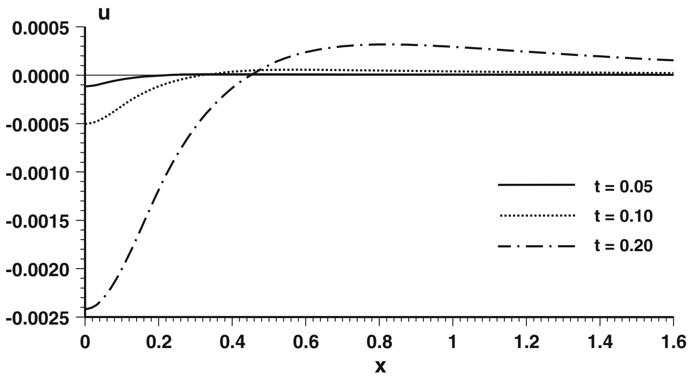

Fig. 2 Displacement distribution in the absence of thermal shock

This integral was evaluated numerically for the different functions using Romberg integration techniques.

5 Numerical Results

Copper material was chosen for purposes of numerical evaluations. The constants of the problem are shown in Table 1

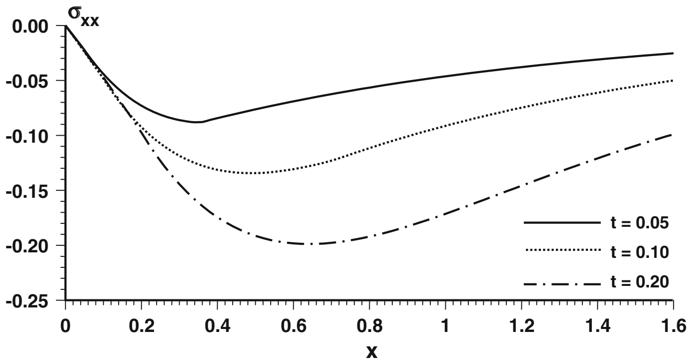

Fig. 3 Stress distribution in the absence of thermal shock

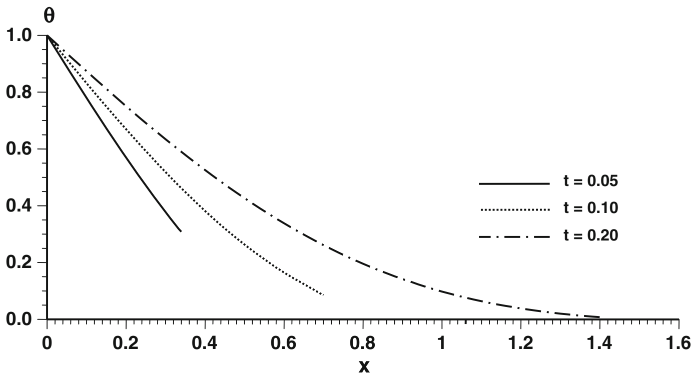

Fig. 4 Temperature distribution in the absence of body forces and heat sources

The computations were performed for three values of non-dimensional time, equal to $t = 0.05$, $t = 0.10$, and $t = 0.20$. The numerical technique outlined above was used to obtain the temperature, displacement, and stress distributions. All functions were evaluated inside the medium on the x-axis ($y = 0$) as functions of x. The temperature increment θ, the displacement component u, and the stress component σ_{xx} are represented by the graphs in Figs. 1, 2, and 3 in the absence of thermal shock and under the effect of the body forces and heat sources. The temperature increment θ, the displacement component u and the stress component σ_{xx} are represented by the graph in Figs. 4, 5, and 6 in the absence of both body forces and heat sources and under the action of the thermal shock. The total effect of body forces, heat sources, and thermal shock are represented in Figs. 7, 8, and 9. We note that due to the symmetry, the displacement component v is identically zero on the x-axis.

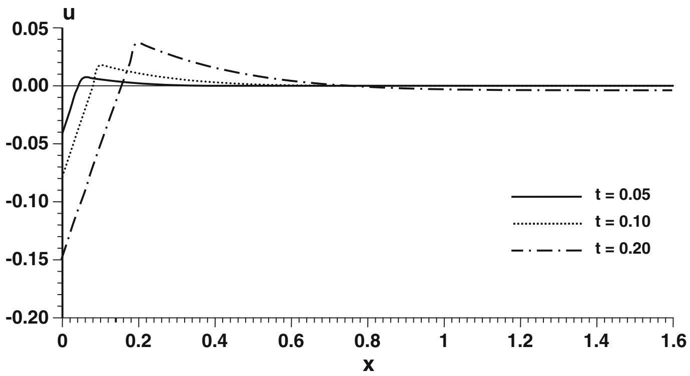

Fig. 5 Displacement distribution in the absence of body forces and heat sources

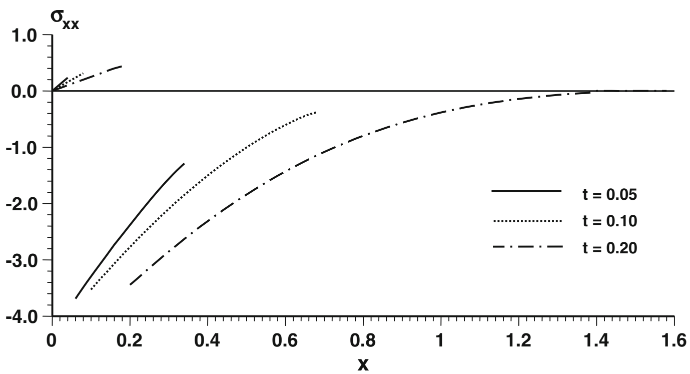

Fig. 6 Stress distribution in the absence of body forces and heat sources

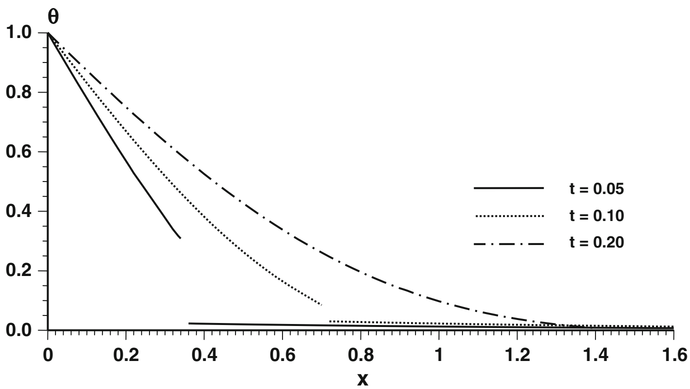

Fig. 7 Total temperature distribution

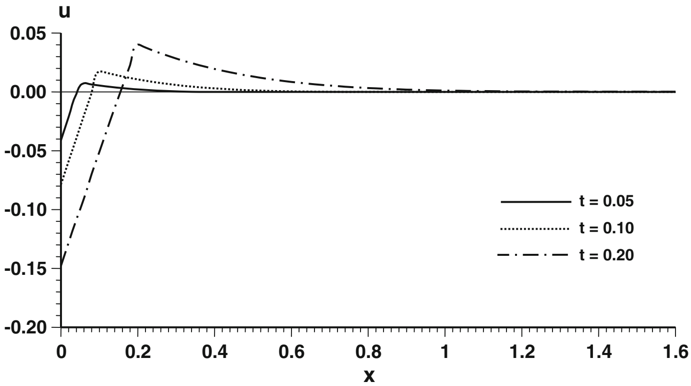

Fig. 8 Total displacement distribution

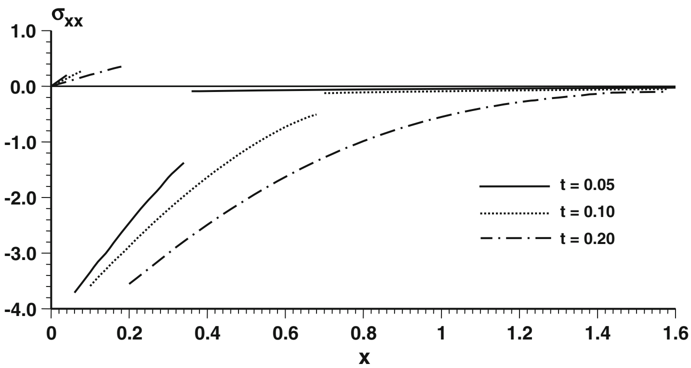

Fig. 9 Total stress distribution

6 Conclusions

The problem is solved using a direct approach that eliminates the well known problems associated with the solution using potential functions. The numerical inversion method proved to be very fast and efficient. The solution exhibits the finite wave speeds associated with the generalized theory of thermoelasticity as opposed to the infinite speeds associated with the coupled theory. The wave fronts for the temperature and stress distributions are represented by discontinuities in the functions themselves and by a discontinuous first derivative in the displacement distribution. The applied force and the heat sources tend to increase all the functions considered.

References

1. H. Lord, Y. Shulman, J. Mech. Phys. Solids **15**, 299 (1967)
2. R. Dhaliwal, H. Sherief, Quart. Appl. Math. **33**, 1 (1980)
3. J. Ignaczak, J. Therm. Stress. **2**, 171 (1979)

4. H. Sherief, R. Dhaliwal, J. Therm. Stress. **3**, 223 (1980)
5. H. Sherief, Quart. Appl. Math. **45**, 773 (1987)
6. M. Anwar, H. Sherief, J. Therm. Stress. **11**, 353 (1988)
7. H. Sherief, J. Therm. Stress. **16**, 163 (1993)
8. H. Sherief, M. Anwar, J. Therm. Stress. **17**, 567 (1994)
9. H. Sherief, J. Therm. Stress. **9**, 151 (1986)
10. H. Sherief, M. Anwar, J. Therm. Stress. **9**, 165 (1986)
11. H. Sherief, N.M. El-Maghraby, J. Therm. Stress. **26**, 333 (2003)
12. H. Sherief, N.M. El-Maghraby, J. Therm. Stress. **28**, 465 (2005)
13. H. Sherief, K. Helmy, J. Therm. Stress. **22**, 897 (1999)
14. N.M. El-Maghraby, J. Therm. Stress. **27**, 227 (2004)
15. N.M. El-Maghraby, J. Therm. Stress. **28**, 1227 (2005)
16. H.A. Saleh, J. Therm. Stress. **28**, 253 (2005)
17. N.M. El-Maghraby, J. Therm. Stress. **32**, 863 (2009)
18. S. Mallik, M. Kanoria, Eur. J. Mech. A **27**, 607 (2008)
19. R.V. Churchill, in *Operational Mathematics*, 3rd edn. (McGraw-Hill, New York, 1972)
20. I.N. Sneddon, in *Fourier Transforms* (Dover Pubs, New York, 1995)
21. G. Honig, U. Hirdes, J. Comp. Appl. Math. **10**, 113 (1984)